Vopěnka's Principle

and Woodin-like cardinals

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Accessible categories and their connections
University of Leeds
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Vopěnka's Principle (VP) is:

1. a large cardinal notion (in particular, a statement which is not provable from ZFC)
2. one of the strongest connections among category theory, model theory and set theory.

VP in category theory

$$\text{Category Theory} \xleftrightarrow{\text{Accessible Categories}} \text{Model Theory}$$

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Theorem

Every accessible category has an accessible full embedding into **Gra***.*

- Objects of **Gra** are arbitrary graphs, i.e. pairs $\langle X, R\rangle$ with $R \subseteq X \times X$.
- Arrows are graph homomorphisms, i.e. functions preserving edges (one-way).

VP in category theory

- A graph $\langle X, R\rangle$ is called rigid if the only homomorphism $f : \langle X, R\rangle \to \langle X, R\rangle$ is the identity.
- More generally, a family of graphs $\{\langle X_i, R_i\rangle \mid i \in I\}$ is called rigid if it admits only the identity morphisms (the full subcategory of **Gra** consisting of these graphs is discrete).

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Theorem (Vopěnka)

For every set X, there is a relation R such that the graph $\langle X, R\rangle$ is rigid.

- **Gra** has rigid objects of any desired size.

VP in category theory

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Corollary

For any cardinal κ, there is a rigid family of graphs $\{\langle X_\alpha, R_\alpha\rangle \mid \alpha < \kappa\}$.

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- **Gra** has rigid families of objects of any desirable size.

VP in category theory

"[...] (Vopěnka) came to the conclusion that, with some more effort, a large rigid class of graphs must surely be also constructible."

(Locally presentable and Accessible Categories, J. Adámek & J. Rosicky, p. 279)

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(Locally presentable and Accessible Categories, J. Adámek & J. Rosicky, p. 279)

Definition (VP - 1st formulation)

Vopěnka's Principle (VP) is the statement that there is no large rigid class of graphs.

VP in category theory

Two of the main category thereotical characterisatons are the following:

Theorem (VP - 2nd formulation)

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Theorem (VP - 3rd formulation)

There is no full embedding $F : \mathbf{Ord} \to \mathbf{Gra}$.

- **Weak Vopěnka's Principle**: There is no full embedding $F : \mathbf{Ord}^{op} \to \mathbf{Gra}$.
- It is known that VP$\Rightarrow$WVP, but still open whether WVP$\Rightarrow$VP.

VP in category theory

Theorem (VP - 2nd formulation)

VP is equivalent to the statement that there is no accessible category with a large rigid class of objects.

- If A and B are structures of the same language, then $j : A \to B$ is called an elementary embedding if for every formula $\phi(v_1, \ldots, v_n)$ and $x_1, \ldots, x_n \in A$, $A \models \phi(x_1, \ldots, x_n) \iff B \models \phi(j(x_1), \ldots, j(x_n))$.
- Suppose T is a first-order theory. The category with objects the models of T and elementary embeddings as morphisms is accessible.

VP in model theory

Theorem (VP - 4th formulation)

Suppose $\{A_\alpha \mid \alpha \in \text{Ord}\}$ *is a class of first-order structures of the same language. Then there are* $\alpha < \beta$ *such that there is an elementary embedding* $j : A_\alpha \to A_\beta$.

VP in model theory

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- Informally: when you have “class” many objects, strong similaritires appear.
- We actually get class many embeddings between the objects (if there are set-many, remove them and consider the new class of structures).
- This formulation easily shows that VP not provable from ZFC.

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- We actually get class many embeddings between the objects (if there are set-many, remove them and consider the new class of structures).
- This formulation easily shows that VP not provable from ZFC.

Proof. Take the structures $\{\langle V_{\alpha+1}, \in, \{\alpha\}\rangle \mid \alpha \in \text{Ord}\}$ and consider an elementary embedding $j : \langle V_{\alpha+1}, \in, \{\alpha\}\rangle \to \langle V_{\beta+1}, \in, \{\beta\}\rangle$. Then j is non-trivial and the least ordinal moved by j can be shown to be measurable. □

VP in model theory

- Elementary embeddings are tightly connected with reflection phenomena.
- Reflection is dual to compactness.

Reflection and compactness I

Suppose L is a language in some logic and σ is a property of L-structures.

Definition

A strongly compact cardinal for σ is a cardinal κ such that for every L-structure A, A has the property σ iff every substructure of A of size less than κ has the property σ.

Definition

A reflection cardinal for σ is a cardinal κ, such that for every L-structure A, if A has the property σ then there is a substructure of A of size less than κ that has the property σ.

VP in model theory

Theorem (VP - 5th formulation, Stavi)

The following are equivalent:

1. *VP.*
2. *For every property of structures that is invariant under isomorphism, there is a reflection cardinal.*
3. *For every property of structures that is invariant under isomorphism, there is a strongly compact cardinal.*

Reflection and Compactness II

Suppose L is a logic.

Definition

A strongly compact cardinal for L is a cardinal κ such that every set of sentences which is κ-satisfiable is itself satisfiable.

Definition

A Löwenheim-Skolem-Tarki (LST) number for L is a cardinal κ such that every structure over some vocabulary has an L-elementary substructure of size $< \kappa$.

Note that these cardinals need not exist in general.

VP in model theory

Theorem (VP - 6th formulation)

The following are equivalent:

1. *VP.*
2. *There is a strongly compact cardinal for every logic.*
3. *There is an LST-number for every logic.*

VP in model theory

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Moral: VP is the ultimate reflection/compactness assertion.

VP in model theory

Corollary (VP - 7th formulation)

The following are equivalent:

1. *VP.*
2. *Every class of structures that is closed under substructures and isomorphic copies, can be axiomatised by a universal sentence in some infinitary logic.*

VP in model theory

Corollary (VP - 7th formulation)

The following are equivalent:

1. *VP.*
2. *Every class of structures that is closed under substructures and isomorphic copies, can be axiomatised by a universal sentence in some infinitary logic.*

Moral: every possible "algebraic" class of structures can be set-axiomatised.

VP in model theory

- Reflection and compactness often appears in the large cardinal hierarchy.
- Large cardinals are usually characterised by the existence of elementary embeddings (cf. V. Gitman's talk).

Large cardinals - part I

More precisely, we use non-trivial elementary embeddings of the form

$$V \xrightarrow{\quad j \quad} M,$$

where V is the set-theoretic universe, M some transitive submodel of V.

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where V is the set-theoretic universe, M some transitive submodel of V.

- Each such embedding has a smallest ordinal κ that is getting moved, denoted by $\text{crit}(j)$, which is a large cardinal.
- M is contained in V, but it cannot be the whole of V (otherwise we get an inconsistency).
- Without stronger assumptions, the best we can get is that M is closed under κ-sequences (${}^{\kappa}M \subseteq M$).

Large cardinals - part I

Definition

A cardinal κ is supercompact if for all $\lambda \geqslant \kappa$, κ is λ-supercompact, i.e. there is an elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and ${}^{\lambda}M \subseteq M$.

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- It can be shown that κ is supercompact iff it is a reflection cardinal for every second-order property in some language of size $< \kappa$.
- VP talks about "every property", so we can add predicates in the definition.

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- VP talks about "every property", so we can add predicates in the definition.

Definition

A cardinal κ is supercompact for A for some class A, if for all $\lambda \geqslant \kappa$, κ is λ-supercompact for A, i.e. there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, ${}^{\lambda}M \subseteq M$ and $A \cap V_\lambda = j(A) \cap V_\lambda$.

Large cardinals - part I

Definition

A cardinal κ is extendible if it is λ-extendible for all $\lambda \geqslant \kappa$, i.e. there is an elementary embedding $j : V_\lambda \to V_\mu$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$.

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- $$\text{Extendible} \quad \underset{\not\Leftarrow}{\Rightarrow} \quad \text{Supercompact}$$
- It can be shown that κ is extendible iff it is a strongly compact cardinal for second-order logic with disjunctions and quantifications of size $< \kappa$.
- VP refers to every possible logic, so we add a predicate again.

Large cardinals - part I

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- VP refers to every possible logic, so we add a predicate again.

Definition

A cardinal κ is extendible for A, for some class A, if for all $\lambda \geqslant \kappa$, κ is λ-extendible for A, i.e. there is an elementary embedding $j : \langle V_\lambda, \in, A \cap V_\lambda \rangle \to \langle V_\mu, \in, A \cap V_\mu \rangle$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \lambda$.

VP in set theory

Theorem (VP - 8th formulation)

The following are equivalent:

1. *VP.*
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3. *For every class A, there is an extendible for A cardinal.*

VP in set theory

Theorem (VP - 8th formulation)

The following are equivalent:

1. *VP.*
2. *For every class A, there is a supercompact for A cardinal.*
3. *For every class A, there is an extendible for A cardinal.*

- This characterisation shows that VP is a large cardinal notion.
- Small caveat: In all formulations so far, we quantify over classes! We either express it as a scheme in ZFC or work in some class theory (but then we get a stronger notion...).
- A solution is to define Vopěnka cardinals.

VP in set theory

Definition

A cardinal δ is a Vopěnka cardinal if for every sequence of structures $\langle A_\alpha \mid \alpha < \delta \rangle$ over some language L of size less than δ, such that $A_\alpha \in V_\delta$ for all α, there is an elementary embedding $j : A_\alpha \to A_\beta$ for some ordinals $\alpha \neq \beta$.

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Theorem

The following are equivalent:

1. *δ is a Vopěnka cardinal.*
2. *For every $A \subseteq V_\delta$, there is a $<\delta$-extendible for A cardinal $\kappa < \delta$.*
3. *For every $A \subseteq V_\delta$, there is a $<\delta$-supercompact for A cardinal $\kappa < \delta$.*

VP in set theory

Theorem

The following are equivalent:

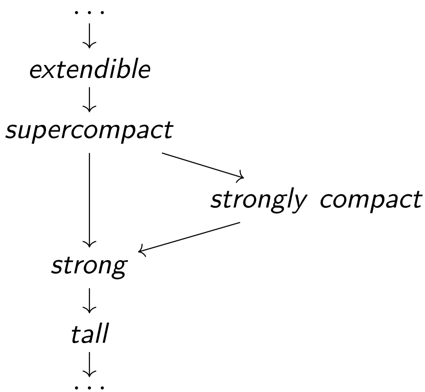
1. δ *is a Vopěnka cardinal.*
2. *For every* $A \subseteq V_\delta$, *there is a* $<\delta$*-extendible for* A *cardinal* $\kappa < \delta$.
3. *For every* $A \subseteq V_\delta$, *there is a* $<\delta$*-supercompact for* A *cardinal* $\kappa < \delta$.

- Recall that supercompact and extendible cardinals are not equivalent.

Question

Why do supercompact and extendible cardinals give the same sort of Vopěnka cardinal? Can we replace them with other large cardinal notions?

Fragment of the large cardinal hierarchy



Large cardinals - part II

Recall: κ is supercompact if for every $\lambda \geqslant \kappa$ there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, ${}^{\lambda}M \subseteq M$.

Large cardinals - part II

Recall: κ is supercompact if for every $\lambda \geqslant \kappa$ there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, ${}^{\lambda}M \subseteq M$.

Definition

A cardinal κ is strong if for every $\lambda \geqslant \kappa$, κ is λ-strong, i.e. there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $V_\lambda \subseteq M$.

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Definition

A cardinal κ is tall if for every $\lambda \geqslant \kappa$, κ is λ-tall, i.e. there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$ and ${}^{\kappa}M \subseteq M$.

Recall: κ is supercompact if for every $\lambda \geqslant \kappa$ there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, ${}^{\lambda}M \subseteq M$.

Definition

A cardinal κ is strong for A, for some set A if for every $\lambda \geqslant \kappa$, κ is λ-strong for A, i.e. there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $V_\lambda \subseteq M$ and $A \cap V_\lambda = j(A) \cap V_\lambda$.

Definition

A cardinal κ is tall for A, for some set A if for every $\lambda \geqslant \kappa$, κ is λ-tall for A, i.e. there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $A \cap \lambda = j(A) \cap \lambda$.

Large cardinals - part II

$$\text{Strong} \quad \Rightarrow \quad \text{Tall}$$
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Theorem

The following are equivalent for a cardinal δ:

1. *For every $A \subseteq V_\delta$ there is a $<\delta$-strong for A cardinal $\kappa < \delta$.*
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A cardinal δ is a Woodin cardinal if one of the previous equivalent conditions holds.

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Proposition

Every Vopěnka cardinal is Woodin and has tons of Woodin cardinals below.

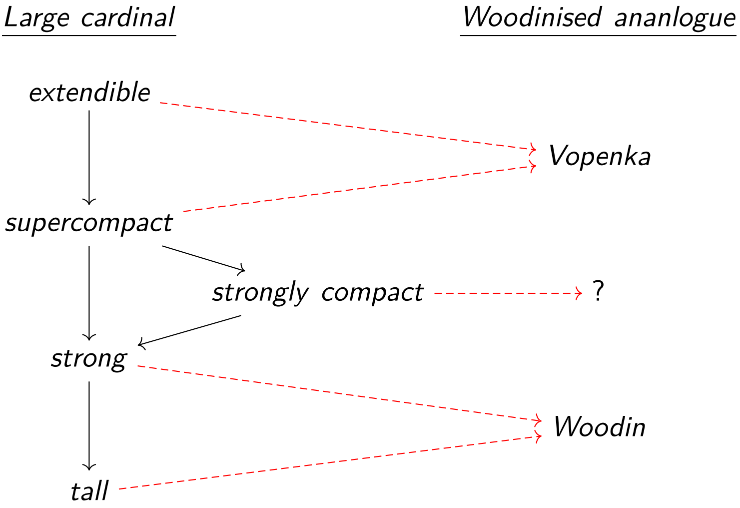
Woodin-like cardinals

A cardinal δ is Woodin for X if:

for every $A \subseteq V_\delta$ there is an X-cardinal with an embedding that reflects A,

- Vopěnka $\equiv$ Woodin for supercompactness $\equiv$ Woodin for extendibility
- Woodin $\equiv$ Woodin for strongness $\equiv$ Woodin for tallness

Woodin-like cardinals



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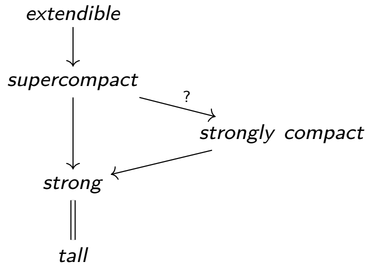
Woodin-like cardinals

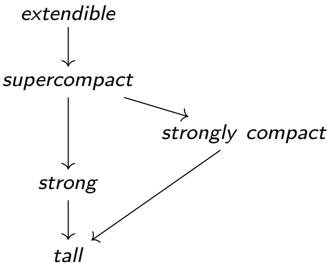
Strong compactness is a **pathological** concept.

- We don't know its exact consistency strength.
- Depending on the models of set theory, it has different large cardinal properties.

Consistency strength



Implication



Woodinised strong compactness

Definition

An infinite cardinal δ is Woodin for strong compactness if for every $A \subseteq \delta$ there is a $<\!\delta$-strongly compact for A cardinal $\kappa < \delta$.

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- Vopěnka $\Rightarrow$ Woodin for strong compactness $\Rightarrow$ Woodin

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- $$\text{Vopěnka} \;\; \underset{\overset{?}{\Leftarrow}}{\Rightarrow} \;\; \text{Woodin for strong compactness} \;\; \underset{\overset{?}{\Leftarrow}}{\Rightarrow} \;\; \text{Woodin}$$

Proposition

If δ is Woodin and a limit of $<\delta$-supercompact cardinals, then it is Woodin for strong compactness.

- There are many such cardinals below a Vopěnka cardinal. Hence, the first implication is strict.

Woodinised strong compactness

Woodin limits of supercompacts	$\Rightarrow$	Woodin for strong compactness	$\Rightarrow$	Woodin

Woodinised strong compactness

Woodin limits of supercompacts ⇒ Woodin for strong compactness ⇒ Woodin

Theorem (D., 2018)

The first Woodin for strong compactness cardinal can consistently be the first Woodin cardinal or the first Woodin limit of supercompact cardinals.

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Theorem (D., 2018)

The first Woodin for strong compactness cardinal can consistently be the first Woodin cardinal or the first Woodin limit of supercompact cardinals.

- Contrast with the following.

Theorem ("Identity crisis", Magidor, '76)

The first strongly compact cardinal can consistently be the first tall or the first supercompact cardinal.

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- The first theorem can be seen as a Woodinised version of the second.

Woodinised strong compactness

Despite the identity crisis, Woodin for strong compactness cardinals have some nice properties:

Woodinised strong compactness

Despite the identity crisis, Woodin for strong compactness cardinals have some nice properties:

Theorem

A cardinal δ is Woodin for strong compactness iff for every function $f : \delta \to \delta$ there is $\kappa < \delta$, which is a closure point of f, and there is an elementary embedding

$$j : V \to M$$

with $\mathrm{crit}(j) = \kappa$, $V_{j(f)(\kappa)} \subseteq M$ and the $j(f)(\kappa)$-covering property. Moreover, j can be assumed to be first-order definable.

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Theorem

There is a naturally defined normal filter on any Woodin for strong compactness cardinal.

Future directions

Question

Can the Woodinised versions of large cardinals give new information about the large cardinal hierarchy?

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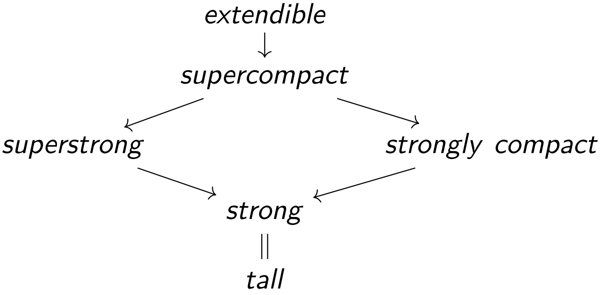

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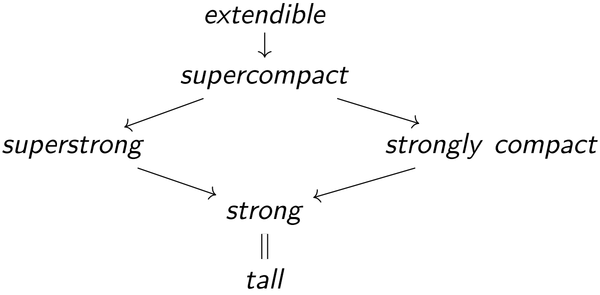

- In particular, what do Woodin for superstrength cardinals look like?

Future directions

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How do the Woodin-like cardinals relate to weakenings of VP?

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Weak Vopěnka's Principle is equivalent to either of these statements:

1. There is no full embedding $F : \mathbf{Ord}^{op} \to \mathbf{Gra}$.
2. There is no class $\langle A_\alpha \mid \alpha \in \mathrm{Ord} \rangle$ of first-order structures such that for all $\alpha < \beta$, there is no homomorphism from A to B and for $\alpha \geqslant \beta$ there is only one homomorphism from A_α to A_β.

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Semi-weak Vopěnka's Principle is equivalent to either of these statements:

1. In any locally presentable category, there is no class of objects $\langle A_\alpha \mid \alpha \in \mathrm{Ord} \rangle$ such that $Hom(A, B) \neq \varnothing$ iff $\alpha \geqslant \beta$.
2. There is no class $\langle A_\alpha \mid \alpha \in \mathrm{Ord} \rangle$ of first-order structures such that $\alpha < \beta$ iff there is no homormophism from A_α to A_β.

We always did feel the same
We just saw it from a different point of view
Tangled up in blue

Bob Dylan. "Tangled up in blue". Blood on the tracks. 1975.

We always did feel the same
We just saw it from a different point of view
Tangled up in blue

Bob Dylan. "Tangled up in blue". Blood on the tracks. 1975.

Thank you!